Some computational results on numbers of the form $p + F_n$

Joint work with Huixi Li

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Christian Goldbach
(1690-1764)

Let

$$\begin{aligned}\mathcal{S}_1 &= \{p + q : p \text{ and } q \text{ are odd primes}\} \\ &= \{6, 8, 10, 12, 14, 16, 18, \cdots\}.\end{aligned}$$

Conjecture (Goldbach, 1742)

The set $\mathcal{S}_1$ contains all even number which are greater than 4.

Émile Lemoine
(1840-1912)

Let

$$\mathcal{S}_2 = \{2p + q : p \text{ and } q \text{ are odd primes}\} = \{9, 11, 13, 15, 17, 19, \cdots\}.$$

Conjecture (Lemoine, 1894)

The set $\mathcal{S}_2$ contains all odd numbers which are greater than 7.

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Some famous conjectures—de Polignac Conjecture

Let

$$\mathcal{S}_3 = \{p + 2^k : p \text{ is an odd prime}, k \geq 1\}$$
$$= \{5, 7, 9, 11, 13, 15, 17, 19, \cdots\}.$$

Conjecture (de Poliganc, 1849)

The set $\mathcal{S}_3$ contains all odd number which are greater than 3.

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Some famous conjectures—de Polignac Conjecture

This conjecture of de Polignac is false. The numbers

$$1, 3, 127, 149, 251, 331, 337, 373, 509, 599, 701,$$
$$757, 809, 877, 905, 907, 959, 977, 997, \cdots$$

cannot be written as a sum of an odd prime and a power of 2.

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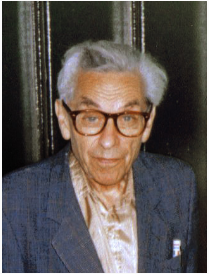

Paul Erdős
(1913-1996)

Theorem (Erdős, 1950)

Numbers in the arithmetic progression

$$7629217 \pmod{11184810}$$

cannot be written as the sum of an odd prime and a power of 2.

History of research on problems related to $p + F_n$

Consider the sum set of the primes and the Fibonacci numbers

$$\mathcal{S}_{p+F_n} = \{m : m \text{ is of the form } p + F_n, n \geq 0\}.$$

- In 2010, Lee proved $\mathcal{S}_{p+F_n}$ has positive lower asymptotic density.
- In 2021, Lee's result was made explicit by Liu and Xue to 0.0254905.
- In 2023, the result was improved to 0.143 by Wang and Chen.

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Question

Can we find an infinite arithmetic progression in the set

$\mathcal{S} = \{m : m \text{ is a positive integer NOT of the form } p + F_n, n \in \mathbb{N}\}$?

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Theorem (Šiurys, 2016)

The arithmetic progression

$$2019544239293395 \pmod{2111872080374430}$$

does not contain integers of the form $p^\alpha \pm F_n$.

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Notice that, the period of the Fibonacci numbers modulo 2111872080374430 is 360.

Our research on problems related to $p + F_n$

We started by checking other moduli with a Pisano period of 360. And we proved the following result.

Proposition

The arithmetic progressions

$$\begin{aligned}
&208641 \pmod{17160990},\\
&218331 \pmod{17160990},\\
&520659 \pmod{17160990},\\
&\cdots \quad \textit{(110 in total)}
\end{aligned}$$

do not contain integers of the form $p + F_n$.

Our research on problems related to $p + F_n$

Soon we proved that there exist arithmetic progressions with common difference of $312018 (= 17160990/55)$.

Theorem

The two arithmetic progressions $208641 \pmod{312018}, 218331 \pmod{312018}$ *do not contain integers of the form* $p + F_n$.

This makes us think whether 312018 is the smallest common difference of arithmetic progressions with this property.

Proof of the minimality of common difference by computation

Because of the modular periodicity of Fibonacci numbers, we can use Dirichlet's theorem to deal with this problem by the method of shifting numbers.

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For example, we know that there exists some prime p such that $p \equiv 1 \pmod{10}$ by Dirichlet's theorem, which satisfies $p + F_1 \equiv p + 1 \equiv 2 \pmod{10}$, that is, the arithmetic progression $10k + 2$ does not satisfy the property we want, because it is not completely in $\mathcal{S}$, where

$$\mathcal{S} = \{m : m \text{ is a positive integer NOT of the form } p + F_n, n \in \mathbb{N}\}.$$

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Proof of the minimality of common difference by computation

For case of 10:

All residues modulo 10:

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9$$

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Proof of the minimality of common difference by computation

For case of 10:

All residues modulo 10:

0	1	2	3	4	5	6	7	8	9

Reduced residues modulo 10:

	1		3				7		9

Shifted by $0 = F_0$:

	1		3				7		9

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Shifted by $2 = F_3$:

	1		3		5				9

Proof of the minimality of common difference by computation

For case of 10:

All residues modulo 10:

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Shifted by $0 = F_0$:

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Shifted by $1 = F_1$:

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Shifted by $2 = F_3$:

	1		3		5				9

Shifted by $3 = F_4$:

0		2		4		6			

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Proof of the minimality of common difference by computation

For case of 10:

All residues modulo 10:

0	1	2	3	4	5	6	7	8	9

Reduced residues modulo 10:

	1		3				7		9

Shifted by $0 = F_0$:

	1		3				7		9

Shifted by $1 = F_1$:

0		2		4				8	

Shifted by $2 = F_3$:

	1		3		5				9

Shifted by $3 = F_4$:

0		2		4		6			

The last 4 rows cover all residues modulo 10, so the modulus 10 can be excluded.

Proof of the minimality of common difference by computation

For case of 10:

All residues modulo 10:

0 1 2 3 4 5 6 7 8 9

Reduced residues modulo 10:

1 3 7 9

Shifted by $0 = F_0$:

	1		3				7		9

Shifted by $1 = F_1$:

0		2		4				8	

Shifted by $2 = F_3$:

	1		3		5				9

Shifted by $3 = F_4$:

0		2		4		6			

The last 4 rows cover all residues modulo 10, so the modulus 10 can be excluded.

In short, for any $a \in \{0, 1, \cdots, 9\}$ and any F_n, if $(10x + a - F_n, 10) = 1$, then there exists a prime number p such that $10x + a - F_n = p$.

Proof of the minimality of common difference by computation

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Huixi Li

Theorem (C.-Li, 2024 Apr.)

Let $\mathcal{S}$ be the set of positive odd integers not of the form $p + F_n$. We have $208641 \pmod{312018}$ and $218331 \pmod{312018}$ are the only two arithmetic progressions in $\mathcal{S}$ with modulus 312018. Moreover, 312018 is the minimal common difference with this property.

The computation time is about 9 hours.

Explanation from the perspective of covering system

We claim that the arithmetic progression $208641 \pmod{312018}$ corresponds to a covering system.

Since the remainders of the Fibonacci numbers modulo 2 are $1, 1, 0, 1, 1, 0, \cdots$ with period 3, we know $n \equiv 1$ or $2 \pmod 3$ is equivalent to $F_n \equiv 1 \pmod 2$.

Therefore, when $x \equiv 1 \pmod 2$ and $n \equiv 1$ or $2 \pmod 3$, we have $x - F_n \equiv 0 \pmod 2$.

Similarly, when $x \equiv 0 \pmod 3$ and $n \equiv 0$ or $4 \pmod 8$, we have $x - F_n \equiv 0 \pmod 3$, etc.

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Explanation from the perspective of covering system

$1, 2$	$\pmod 3$	1	$\pmod 2$
$0, 4$	$\pmod 8$	0	$\pmod 3$
$7, 9, 10, 14$	$\pmod{16}$	6	$\pmod 7$
$0, 9, 18, 27$	$\pmod{36}$	0	$\pmod{17}$
$3, 8, 15$	$\pmod{18}$	2	$\pmod{19}$
$6, 18$	$\pmod{48}$	8	$\pmod{23}$

For every term $x \in 208641 \pmod{312018}$ and every $n \geq 0$, we have $x - F_n$ is divisible by some prime $p \in \{2, 3, 7, 17, 19, 23\}$.

This shows the arithmetic progression $208641 \pmod{312018}$ corresponds to the covering system constructed above.

Explanation from the perspective of covering system

The covering system corresponding to the arithmetic progression $218331 \pmod{312018}$ is similar to that of $208641 \pmod{312018}$, so we just present the following table and omit the explanation.

$1, 2$	$\pmod{3}$	1	$\pmod{2}$
$0, 4$	$\pmod{8}$	0	$\pmod{3}$
$1, 2, 6, 15$	$\pmod{16}$	1	$\pmod{7}$
$0, 9, 18, 27$	$\pmod{36}$	0	$\pmod{17}$
$3, 8, 15$	$\pmod{18}$	2	$\pmod{19}$
$30, 42$	$\pmod{48}$	15	$\pmod{23}$

A conjecture and thank you

Conjecture

Any arithmetic progression with the property can be explained by covering system (using the way we mentioned above).

Thanks!

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