Applications of Decoupling Inequality in Vinogradov Problems and Discrete Strichartz Estimates

HADES Seminar

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Nankai/UC Berkeley

Vinogradov Problem

Let $s \geq 1$ be an integer and N be a large positive integer.

Consider the system of Diophantine equations:

$$\begin{aligned} x_1 + \cdots + x_s &= x_{s+1} + \cdots + x_{2s} \\ x_1^2 + \cdots + x_s^2 &= x_{s+1}^2 + \cdots + x_{2s}^2 \end{aligned}$$

where $x_1, \ldots, x_{2s}$ are integer variables taking values in $\{0, 1, \ldots, N\}$.

The system above is often referred to as the **quadratic Vinogradov's system**.

If $(x_1, \ldots, x_{2s})$ satisfies the equations, it is a **solution**. Let $\mathcal{N}_s(N)$ denote the number of solutions to this system.

Simple cases

Case $s = 1$: The system becomes $x_1 = x_2$ and $x_1^2 = x_2^2$. It is elementary to see that:

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$$\begin{aligned} x_1 + x_2 &= x_3 + x_4 \\ x_1^2 + x_2^2 &= x_3^2 + x_4^2 \end{aligned}$$

If (x_1, x_2, x_3, x_4) is a solution, then we know that $(x_1, x_1^2), (x_2, x_2^2), (x_3, x_3^2), (x_4, x_4^2)$ form a parallelogram, which implies:

$$(x_1, x_2) = (x_3, x_4) \quad \text{or} \quad (x_1, x_2) = (x_4, x_3)$$

Therefore:

$$\mathcal{N}_2(N) = 2(N + 1)^2 - (N + 1)$$

Solutions where $(x_1, \ldots, x_s)$ is a permutation of $(x_{s+1}, \ldots, x_{2s})$ are called **diagonal solutions**.

Non-Diagonal Solutions ($s \geq 3$)

For $s = 1, 2$, there are only diagonal solutions. This changes starting from $s = 3$.

Example ($s = 3$):

$$0 + 3 + 3 = 1 + 1 + 4$$
$$0^2 + 3^2 + 3^2 = 1^2 + 1^2 + 4^2$$

Therefore, $(0, 3, 3, 1, 1, 4)$ is **not** a diagonal solution.

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Therefore, $(0, 3, 3, 1, 1, 4)$ is **not** a diagonal solution.

Theorem

For every positive integer $s \geq 1$ and every positive integer N, we have

$$\mathcal{N}_s(N) \lesssim_{\epsilon, s} N^{s+\epsilon} + N^{2s-3+\epsilon}$$

for every $\epsilon > 0$.

Exponential Sum Formulation

In fact, we have a stronger theorem.

Theorem

For every $p \geq 2$ and positive integer N, we have

$$\left\| \sum_{n=0}^{N} e^{2\pi i \left(nx + n^2 y \right)} \right\|_{L^p([0,1]^2)} \lesssim_{p,\epsilon} N^{\frac{1}{2}+\epsilon} + N^{1-\frac{3}{p}+\epsilon},$$

for every $\epsilon > 0$.

First, we will show how to derive $\mathcal{N}_s(N) \lesssim_{\epsilon,s} N^{s+\epsilon} + N^{2s-3+\epsilon}$ (the previous theorem) from this theorem, with $p = 2s$.

The Equality of Two Quantities

Consider the 2s-th power of the L^{2s} norm:

$$\left\| \sum_{n=0}^{N} e^{2\pi i \left(nx+n^2y\right)} \right\|_{L^{2s}([0,1]^2)}^{2s} = \iint_{[0,1]^2} \left| \sum_{n=0}^{N} e^{2\pi i \left(nx+n^2y\right)} \right|^{2s} dxdy$$

Expanding the 2s power on the right hand side, we obtain:

$$\iint_{[0,1]^2} \sum_{0\leq n_1,\ldots,n_{2s}\leq N} e^{2\pi i(\star)x+2\pi i(\diamond)y} dxdy$$

where the phase terms are given by:

$$(\star) := n_1+\cdots+n_s-n_{s+1}-\cdots-n_{2s},$$
$$(\diamond) := n_1^2+\cdots+n_s^2-n_{s+1}^2-\cdots-n_{2s}^2.$$

The Equality of Two Quantities

Recall the orthogonality of complex exponentials over $[0,1]$:

$$\int_0^1 e^{2\pi ikt}dt = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0, k \in \mathbb{Z} \end{cases}$$

Applying this to both the x and y integrals, the term inside the sum integrates to 1 when $(\star) = 0$ and $(\diamond) = 0$ simultaneously, and 0 otherwise.

These conditions are exactly the equations of the quadratic Vinogradov's system. Therefore, the integral simply counts the number of solutions:

$$\left\| \sum_{n=0}^{N} e^{2\pi i\left(nx+n^2y\right)} \right\|_{L^{2s}([0,1]^2)}^{2s} = \mathcal{N}_s(N)$$

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According to the result above, we only need to show, for every $p \geq 2$ and positive integer N, we have $\left\|\sum_{n=0}^{N} e^{2\pi i\left(nx+n^2y\right)}\right\|_{L^{p}([0,1]^2)} \lesssim_{p,\epsilon} N^{\frac{1}{2}+\epsilon} + N^{1-\frac{3}{p}+\epsilon}$, for every $\epsilon > 0$.

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We will prove it by a harmonic analysis tool: **Fourier Decoupling Inequality**.

Introduction of Decoupling Inequality (for parabola)

We will introduce the decoupling inequality for parabola first. We fix $\delta \in 2^{-\mathbb{N}}$, i.e. $\delta = 2^{-k}$ for some non-negative integer k. For an interval $I \subset [0,1]$ define the frequency region

$$\Omega_I(\delta) := \left\{ \left(\xi_1, \xi_1^2 + \delta'\right) : \xi_1 \in I, \ |\delta'| \leq \delta^2 \right\}.$$

Let $p \in (1, \infty)$ and let $F : \mathbb{R}^2 \to \mathbb{C}$ be an L^p function whose Fourier transform is supported on $\Omega_{[0,1]}(\delta)$. For each interval I we denote by F_I the Fourier restriction of F to $\Omega_I(\delta)$, i.e.

$$F_I := \mathcal{F}^{-1}\left(\widehat{F} \cdot \mathbf{1}_{\Omega_I(\delta)}\right).$$

(Because $1 < p < \infty$, the L^p boundedness of the Hilbert transform together with Fubini's theorem guarantees that each F_I is again an $L^p(\mathbb{R}^2)$ function.)

Fourier Decoupling Inequality

Let $\mathcal{I}_\delta$ be the collection of all dyadic subintervals $I \subset [0,1]$ of length $\ell(I) = \delta$. The sets $\{\Omega_I(\delta)\}_{I\in\mathcal{I}_\delta}$ form a partition of $\Omega_{[0,1]}(\delta)$ (up to boundaries), and we have:

$$F = \sum_{I\in\mathcal{I}_\delta} F_I.$$

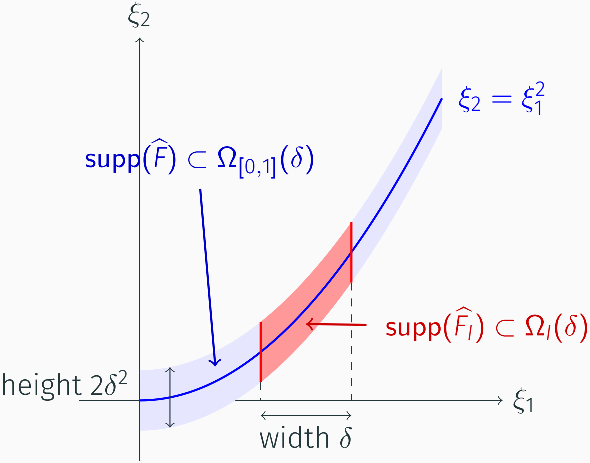
By the triangle inequality, the Cauchy–Schwarz inequality, and noting that $|\mathcal{I}_\delta| = \frac{1}{\delta}$, we obtain the trivial bound:

$$\|F\|_{L^p} \leq \sum_{I\in\mathcal{I}_\delta} \|F_I\|_{L^p} \leq \left(\frac{1}{\delta}\right)^{\frac{1}{2}} \left(\sum_{I\in\mathcal{I}_\delta} \|F_I\|_{L^p}^2\right)^{\frac{1}{2}}$$

Define $D_p(\delta)$ to be the infimum of D such that $\|F\|_{L^p(\mathbb{R}^2)} \leq D \left(\sum_{I\in\mathcal{I}_\delta} \|F_I\|_{L^p(\mathbb{R}^2)}^2\right)^{\frac{1}{2}}$ holds for every F. Choosing F supported on a single $\Omega_I(\delta)$ gives $D_p(\delta) \geq 1$. Thus:

$$1 \leq D_p(\delta) \leq \left(\frac{1}{\delta}\right)^{\frac{1}{2}}$$

Geometric Setup of Decoupling



Fourier Decoupling Inequality

$$\|F\|_{L^p(\mathbb{R}^2)} \leq D_p(\delta) \left(\sum_{I \in \mathcal{I}_\delta} \|F_I\|^2_{L^p(\mathbb{R}^2)} \right)^{\frac{1}{2}}$$

The Main Decoupling Proposition

An inequality of the form

$$\|F\|_{L^p} \leq D_p(\delta) \left(\sum_{I \in \mathcal{I}_\delta} \|F_I\|_{L^p}^2 \right)^{\frac{1}{2}}$$

is called a **Fourier decoupling inequality**. The following proposition gives bounds on $D_p(\delta)$.

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Proposition (Decoupling inequality for parabola)

For every $p \in [2, \infty)$ and every $\varepsilon > 0$, there exists a constant $C_{p,\varepsilon}$ depending only on p and ε such that for all $\delta \in 2^{-\mathbb{N}}$,

$$D_p(\delta) \leq \begin{cases} C_{p,\varepsilon}\, \delta^{-\varepsilon}, & 2 \leq p \leq 6, \\ C_{p,\varepsilon}\, \delta^{-\left(\frac{1}{2} - \frac{3}{p}\right) - \varepsilon}, & p > 6. \end{cases}$$

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In fact, this proposition holds for every $(t, \varphi(t))$ as long as $\varphi''(t) \neq 0$.

Proof of Theorem on the Number of Solutions: Constructing the Function

Assume $N \in 2^{\mathbb{N}}$ and let $\delta := \frac{1}{N}$.

Let χ_0 be a non-negative smooth function supported on $B(0,2) \subset \mathbb{R}^2$, with $\chi_0 = 1$ on $B(0,1)$. We apply decoupling inequality to F given by:

$$\widehat{F}(\xi_1, \xi_2) = \sum_{n=0}^{N} \chi_0 \left(2^{10} N^2 \left(\xi_1 - \frac{n}{N} \right), 2^{10} N^2 \left(\xi_2 - \frac{n^2}{N^2} \right) \right) := \sum_{n=0}^{N} \widehat{F}_n(\xi_1, \xi_2)$$

Taking the inverse Fourier transform, we compute $F = \sum_{n=0}^{N} F_n$, yielding:

$$\sum_{n=0}^{N} e^{i\left(x_1 \frac{n}{N} + x_2 \frac{n^2}{N^2}\right)} \iint_{\mathbb{R}^2} \chi_0 \left(2^{10} N^2 \xi_1, 2^{10} N^2 \xi_2\right) e^{i(x_1\xi_1 + x_2\xi_2)} d\xi_1 d\xi_2$$

Note: The support centers $\frac{n}{N}$ closely align with our interval $[0,1]$.

Proof of Theorem on the Number of Solutions: Lower Bound

Notice that, for $|x_1| \leq \frac{100}{\delta^2}$, $|x_2| \leq \frac{100}{\delta^2}$, the integral is $\geq 2^{-1000}\delta^4$.

Thus, calculating the L^p norm of F:

$$\|F\|_{L^p} \geq 2^{-1000}\delta^4 \left(\int_0^{100N^2} \int_0^{100N^2} \left| \sum_{n=0}^{N} e^{i\left(x_1 \frac{n}{N} + x_2 \frac{n^2}{N^2}\right)} \right|^p dx_1 dx_2 \right)^{\frac{1}{p}}$$

Applying the change of variables $\frac{x_1}{N} \to 2\pi x_1$, $\frac{x_2}{N^2} \to 2\pi x_2$ and using periodicity in the x_1 direction:

$$\|F\|_{L^p} \geq 2^{-1000}\delta^4 N^{\frac{4}{p}} \left(\int_0^1 \int_0^1 \left| \sum_{n=0}^{N} e^{2\pi i\left(n x_1 + n^2 x_2\right)} \right|^p dx_1 dx_2 \right)^{\frac{1}{p}}$$

Proof of Theorem on the Number of Solutions: Upper Bound

To apply decoupling strictly within $\Omega_{[0,1]}(\delta)$, we separate the boundary terms:

$$\|F\|_{L^p} \leq \left\| \sum_{n=1}^{N-1} F_n \right\|_{L^p} + \|F_0\|_{L^p} + \|F_N\|_{L^p}$$

Applying decoupling to the main sum: $\left\| \sum_{n=1}^{N-1} F_n \right\|_{L^p} \leq D_p(\delta) \left(\sum_{n=1}^{N-1} \|F_n\|_{L^p}^2 \right)^{\frac{1}{2}}$.

Since $\|F_n\|_{L^p} \lesssim_{\chi_0} \delta^4 N^{\frac{4}{p}}$ and $D_p(\delta) \geq 1$, we obtain:

$$\|F\|_{L^p} \lesssim_{\chi_0} D_p(\delta)\delta^4 N^{\frac{4}{p}} N^{\frac{1}{2}} + \delta^4 N^{\frac{4}{p}} \lesssim_{\chi_0} D_p(\delta)\delta^4 N^{\frac{4}{p}} N^{\frac{1}{2}}$$

Combining the upper and lower bounds (canceling $\delta^4 N^{\frac{4}{p}}$):

$$\left(\int_0^1 \int_0^1 \left| \sum_{n=0}^{N} e^{2\pi i \left(n x_1 + n^2 x_2 \right)} \right|^p dx_1 dx_2 \right)^{\frac{1}{p}} \lesssim_{\chi_0} N^{\frac{1}{2}} D_p(\delta)$$

Substituting the bound for $D_p(\delta)$ finishes the proof.

Origin of Discrete Strichartz Estimates

Classical discrete Strichartz estimates arise naturally in the study of periodic solutions to dispersive partial differential equations. Throughout this talk, we identify the torus $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$.

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A quintessential example is the initial value problem for the linear Schrödinger equation on the space-time torus $\mathbb{T} \times \mathbb{T}$:

$$i\partial_t u(x,t) - \frac{1}{2\pi}\partial_x^2 u(x,t) = 0, \quad (x,t) \in \mathbb{T} \times \mathbb{T},$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{T}.$$

Discrete Strichartz Estimate for Schrödinger Equation

By taking the Fourier transform in the spatial variable x, the n-th spatial Fourier coefficient $\widehat{u}(n, t)$ of the solution satisfies the ordinary differential equation:

$$i\partial_t \widehat{u}(n, t) + 2\pi n^2 \widehat{u}(n, t) = 0.$$

Solving this yields $\widehat{u}(n, t) = \widehat{u}(n, 0)e^{2\pi i n^2 t}$, where $\widehat{u}(n, 0)$ is the n-th Fourier coefficient of the initial data $u_0(x)$.

Consequently, the full solution to the periodic Schrödinger equation can be expressed as the exponential sum:

$$u(x, t) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i (nx + n^2 t)}.$$

Discrete Strichartz Estimate for Schrödinger Equation

In practice, we are often interested in frequency-localized initial data. Suppose $u_0(x) = \sum_{n=0}^{N} a_n e^{2\pi inx}$ for some integer $N \geq 1$.

By Plancherel's theorem, the L^2 norm of the initial data is given by:

$$\|u_0\|_{L^2(\mathbb{T})} = \left(\sum_{n=0}^{N} |a_n|^2 \right)^{\frac{1}{2}} .$$

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A central problem in partial differential equations and harmonic analysis is understanding the L^p integrability of the solution $u(x,t)$ in terms of the L^2 norm of its initial data.

Discrete Strichartz Estimate for Schrödinger Equation

This central problem is elegantly answered by the foundational discrete Strichartz estimate, famously established by Bourgain's groundbreaking 1993 paper using purely arithmetic methods.

Theorem (Bourgain, 1993)

Let $N \geq 1$ be an integer and $\varepsilon > 0$. Suppose $u_0 \in L^2(\mathbb{T})$ is frequency-localized such that $u_0(x) = \sum_{n=0}^{N} a_n e^{2\pi inx}$. Then for every $p \geq 2$, the corresponding solution $u(x,t)$ to the periodic linear Schrödinger equation satisfies the spacetime bound:

$$\|u\|_{L^p(\mathbb{T}^2)} \leq C_{p,\varepsilon} N^{\varepsilon} \left(1 + N^{\frac{1}{2}-\frac{3}{p}}\right) \|u_0\|_{L^2(\mathbb{T})},$$

where $C_{p,\varepsilon}$ is a positive constant depending only on p and ε.

Discrete Strichartz Estimates and Vinogradov's Problem

Equivalently, written purely in terms of the Fourier coefficients, one has the classical exponential sum estimate:

$$\left\| \sum_{n=0}^{N} a_n e^{2\pi i (nx+n^2 t)} \right\|_{L^p([0,1]^2)} \lesssim_{p,\varepsilon} N^{\varepsilon} \left(1 + N^{\frac{1}{2} - \frac{3}{p}} \right) \left(\sum_{n=0}^{N} |a_n|^2 \right)^{\frac{1}{2}} .$$

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Connection to Vinogradov's Problem: This discrete Strichartz estimate is deeply connected to analytic number theory. If we let $a_n = 1$ for all $0 \leq n \leq N$, the left hand side becomes the L^p norm of the summation $\sum_{n=0}^{N} e^{2\pi i(nx+n^2t)}$. In this case, the bound directly recovers the classical estimates for Vinogradov's mean value theorem in degree 2.

Proof of Discrete Strichartz Estimates: Constructing the Function

Assume $N \in 2^{\mathbb{N}}$ and let $\delta := \frac{1}{N}$.

Let χ_0 be a non-negative smooth function supported on $B(0,2) \subset \mathbb{R}^2$, with $\chi_0 = 1$ on $B(0,1)$. We apply our estimates to $F = \sum_{n=0}^{N} F_n$, where:

$$\widehat{F}(\xi_1, \xi_2) = \sum_{n=0}^{N} a_n \chi_0 \left(2^{10} N^2 \left(\xi_1 - \frac{n}{N} \right), 2^{10} N^2 \left(\xi_2 - \frac{n^2}{N^2} \right) \right)$$

Taking the inverse Fourier transform and applying the change of variables $\xi_1 - \frac{n}{N} \to \xi_1, \xi_2 - \frac{n^2}{N^2} \to \xi_2$, we compute $F(x,t)$ yielding:

$$\sum_{n=0}^{N} a_n e^{i\left(x\frac{n}{N} + t\frac{n^2}{N^2}\right)} \iint_{\mathbb{R}^2} \chi_0 \left(2^{10} N^2 \xi_1, 2^{10} N^2 \xi_2\right) e^{i(x\xi_1 + t\xi_2)} d\xi_1 d\xi_2$$

Proof of Discrete Strichartz Estimates: Lower Bound

Note that for $|x| \leq \frac{100}{\delta^2}, |t| \leq \frac{100}{\delta^2}$, the inner integral is $\geq 2^{-1000}\delta^4$.

Thus, calculating the L^p norm of F:

$$\|F\|_{L^p(\mathbb{R}^2)} \geq 2^{-1000}\delta^4 \left(\int_0^{100N^2} \int_0^{100N^2} \left| \sum_{n=0}^{N} a_n e^{i\left(x\frac{n}{N}+t\frac{n^2}{N^2}\right)} \right|^p dxdt \right)^{\frac{1}{p}}$$

Applying the change of variables $\frac{x}{N} \to 2\pi x$, $\frac{t}{N^2} \to 2\pi t$ and using periodicity in the x direction:

$$\|F\|_{L^p(\mathbb{R}^2)} \geq 2^{-1000}\delta^4 N^{\frac{4}{p}} \left(\int_0^{1} \int_0^{1} \left| \sum_{n=0}^{N} a_n e^{2\pi i\left(nx+n^2t\right)} \right|^p dxdt \right)^{\frac{1}{p}}$$

Proof of Discrete Strichartz Estimates: Upper Bound

Applying triangle inequality for boundary terms and decoupling for parabola to the main sum:

$$\|F\|_{L^p(\mathbb{R}^2)} \leq D_p(\delta) \left(\sum_{n=1}^{N-1} \|F_n\|_{L^p(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}} + \|F_0\|_{L^p(\mathbb{R}^2)} + \|F_N\|_{L^p(\mathbb{R}^2)}$$

Using $\|F_n\|_{L^p(\mathbb{R}^2)} \lesssim_{\chi_0} |a_n| \delta^4 N^{\frac{4}{p}}$ and $D_p(\delta) \geq 1$, we simplify the upper bound:

$$\|F\|_{L^p(\mathbb{R}^2)} \lesssim_{\chi_0} D_p(\delta) \delta^4 N^{\frac{4}{p}} \left(\sum_{n=0}^{N} |a_n|^2 \right)^{\frac{1}{2}}$$

Combining this with the lower bound (canceling $\delta^4 N^{\frac{4}{p}}$):

$$\left(\int_0^1 \int_0^1 \left| \sum_{n=0}^{N} a_n e^{2\pi i \left(nx + n^2 t \right)} \right|^p dxdt \right)^{\frac{1}{p}} \lesssim_{\chi_0} D_p(\delta) \left(\sum_{n=0}^{N} |a_n|^2 \right)^{\frac{1}{2}}$$

Substituting $D_p(\delta) \lesssim_{p,\varepsilon} N^{\varepsilon} \left(1 + N^{\frac{1}{2} - \frac{3}{p}} \right)$ finishes the proof.

Compared with Efficient Congruencing Method

There are two primary approaches to addressing problems involving discrete Strichartz estimates: decoupling and efficient congruencing. Furthermore, among currently known various discrete Strichartz estimates, the best bounds have been achieved either through decoupling methods or via efficient congruencing. The latter is a number-theoretic approach pioneered by Wooley across a series of publications in 2010s.

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$$\left\|\sum_{n=1}^{N} a_n e^{2\pi i\left(\boxed{g(n)}x+\boxed{h(n)}t\right)}\right\|_{L^p([0,1]^2)} \lesssim_{p,\varepsilon} N^{\boxed{E_p}+\varepsilon}\left(\sum_{n=1}^{N}|a_n|^2\right)^{\frac{1}{2}}.$$

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$$\sum_{n=0}^{N} a_n e^{2\pi i(nx+n^2t)} \qquad \sum_{n=0}^{N} a_n e^{2\pi i(nx+n^{1+\nu}t)}$$

$$\sum_{n=0}^{N} a_n e^{2\pi i(nx+n^3t)} \qquad \sum_{n=0}^{N} a_n e^{2\pi i(n^{1+\mu}x+n^{1+\nu}t)}$$

$$\sum_{n=0}^{N} a_n e^{2\pi i(\mathrm{Poly}_{k_1}(n)x+\mathrm{Poly}_{k_2}(n)t)} \qquad \sum_{n=0}^{N} a_n e^{2\pi i(nx+(n^k+O(1))t)}$$

Compared with Efficient Congruencing Method

There are two primary approaches to addressing problems involving discrete Strichartz estimates: decoupling and efficient congruencing. Furthermore, among currently known various discrete Strichartz estimates, the best bounds have been achieved either through decoupling methods or via efficient congruencing. The latter is a number-theoretic approach pioneered by Wooley across a series of publications in 2010s.

$$\left\|\sum_{n=1}^{N} a_n e^{2\pi i(\boxed{g(n)}x+\boxed{h(n)}t)}\right\|_{L^p([0,1]^2)} \lesssim_{p,\varepsilon} N^{\boxed{E_p}+\varepsilon}\left(\sum_{n=1}^{N}|a_n|^2\right)^{\frac{1}{2}}.$$

$\sum_{n=0}^{N} a_n e^{2\pi i(nx+n^2t)}$	$\sum_{n=0}^{N} a_n e^{2\pi i(nx+n^{1+\nu}t)}$
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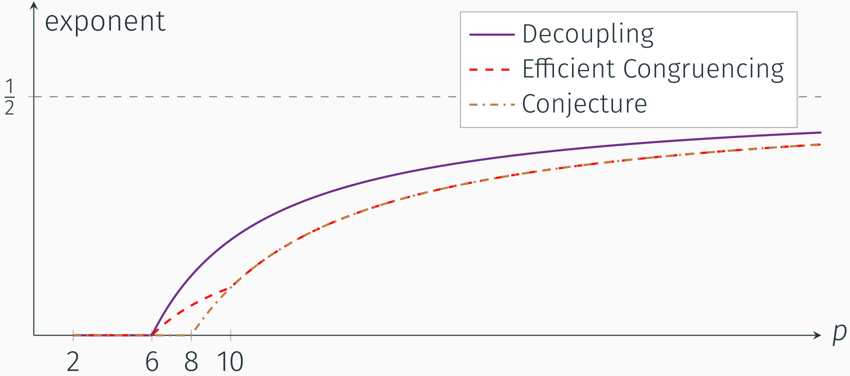
Discrete Strichartz Estimates for KdV Equation

Theorem (Wooley, 2022)

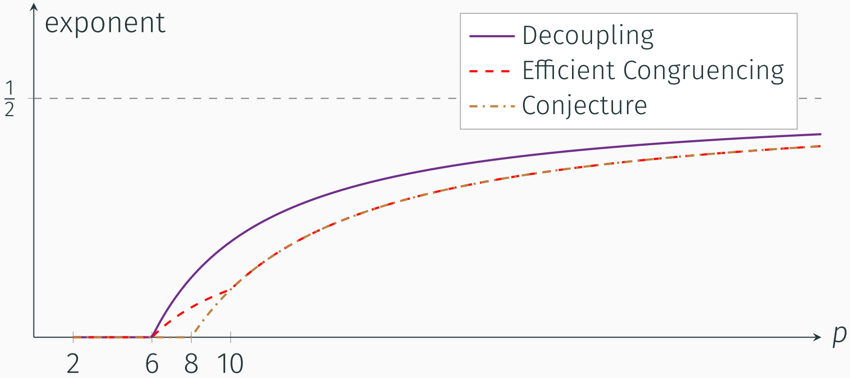
$$\left\| \sum_{n=0}^{N} a_n e^{2\pi i(nx+n^3t)} \right\|_{L^p([0,1]^2)} \lesssim_{p,\varepsilon} N^{\varepsilon} \left(1 + N^{\frac{1}{2}-\frac{4}{p}}\right) \left(\sum_{n=0}^{N} |a_n|^2 \right)^{\frac{1}{2}} \text{ for } p \in [2,6] \cup [10,\infty).$$



Discrete Strichartz Estimates for KdV Equation

Conclusion by Decoupling (C., 2026+)

$$\left\|\sum_{n=0}^{N} a_n e^{2\pi i(nx+n^3 t)}\right\|_{L^p([0,1]^2)} \lesssim_{p,\varepsilon} N^{\varepsilon}\left(1+N^{\frac{1}{2}-\frac{3}{p}}\right)\left(\sum_{n=0}^{N}|a_n|^2\right)^{\frac{1}{2}}.$$



Refined Decoupling Inequality

Let $\Omega_I^{(\nu)}(\delta) := \left\{\left(\xi_1, \xi_1^{1+\nu} + \delta'\right) : \xi_1 \in I,\ |\delta'| \leq \delta^r\right\}$, $F_I := \mathcal{F}^{-1}\left(\widehat{F} \cdot \mathbf{1}_{\Omega_I^{(\nu)}(\delta)}\right)$. Suppose $F = \sum_{I \in \mathcal{I}_\delta} F_I$, we define $D_p^{(\nu)}(\delta)$ to be the infimum of D such that $\|F\|_{L^p(\mathbb{R}^2)} \leq D \left(\sum_{I \in \mathcal{I}_\delta} \|F_I\|_{L^p(\mathbb{R}^2)}^2\right)^{\frac{1}{2}}$.

Proposition (Decoupling inequality for $(t, t^{1+\nu})$, Biswas et al., 2020)

For every fixed $\nu > 0$, let $r = \max\{1 + \nu, 2\}$. For every $p \in [2, 6]$ and every $\varepsilon > 0$, there exists a constant $C_{p,\varepsilon}$ depending only on p and ε such that for all $\delta \in 2^{-\mathbb{N}}$,

$$D_p^{(\nu)}(\delta) \leq C_{p,\varepsilon}\, \delta^{-\varepsilon}.$$

Proof idea: The core of the proof lies in exactly decomposing the unit interval $[0, 1]$ into a small neighborhood $[0, \delta^{\frac{1}{2}-\epsilon}]$ to isolate the singularity where the curvature vanishes, alongside $O(\log \delta^{-1})$ dyadic intervals $[2^{-k}, 2^{-k+1}]$ that completely cover the remaining interval $[\delta^{\frac{1}{2}-\epsilon}, 1]$.

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Proof of Discrete Strichartz Estimates for KdV Equation: Constructing the Function

We only need to show it's true for $p = 2, 6$.

Assume $N \in 2^{\mathbb{N}}$ and let $\delta := \frac{1}{N}$.

Let χ_0 be a non-negative smooth function supported on $B(0,2) \subset \mathbb{R}^2$, with $\chi_0 = 1$ on $B(0,1)$. We apply our estimates to $F = \sum_{n=0}^{N} F_n$, where:

$$\widehat{F}(\xi_1, \xi_2) = \sum_{n=0}^{N} a_n \chi_0 \left(2^{10} N^3 \left(\xi_1 - \frac{n}{N} \right), 2^{10} N^3 \left(\xi_2 - \frac{n^3}{N^3} \right) \right)$$

Taking the inverse Fourier transform and applying the change of variables $\xi_1 - \frac{n}{N} \to \xi_1, \xi_2 - \frac{n^3}{N^3} \to \xi_2$, we compute $F(x,t)$ yielding:

$$\sum_{n=0}^{N} a_n e^{i\left(x\frac{n}{N} + t\frac{n^3}{N^3}\right)} \iint_{\mathbb{R}^2} \chi_0 \left(2^{10} N^3 \xi_1, 2^{10} N^3 \xi_2\right) e^{i(x\xi_1 + t\xi_2)} d\xi_1 d\xi_2$$

Proof of Discrete Strichartz Estimates for KdV Equation: Lower Bound

Note that for $|x| \leq \frac{100}{\delta^3}, |t| \leq \frac{100}{\delta^3}$, the inner integral is $\geq 2^{-1000}\delta^6$.

Thus, calculating the L^p norm of F:

$$\|F\|_{L^p(\mathbb{R}^2)} \geq 2^{-1000}\delta^6 \left(\int_0^{100N^3} \int_0^{100N^3} \left| \sum_{n=0}^{N} a_n e^{i\left(x\frac{n}{N}+t\frac{n^3}{N^3}\right)} \right|^p dxdt \right)^{\frac{1}{p}}$$

Applying the change of variables $\frac{x}{N} \to 2\pi x$, $\frac{t}{N^3} \to 2\pi t$ and using periodicity in the x direction:

$$\|F\|_{L^p(\mathbb{R}^2)} \geq 2^{-1000}\delta^6 N^{\frac{6}{p}} \left(\int_0^{1} \int_0^{1} \left| \sum_{n=0}^{N} a_n e^{2\pi i\left(nx+n^3t\right)} \right|^p dxdt \right)^{\frac{1}{p}}$$

Proof of Discrete Strichartz Estimates for KdV Equation: Upper Bound

Applying triangle inequality for boundary terms and decoupling for (t, t^3) to the main sum:

$$\|F\|_{L^p(\mathbb{R}^2)} \leq D_p^{(2)}(\delta) \left(\sum_{n=1}^{N-1} \|F_n\|_{L^p(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}} + \|F_0\|_{L^p(\mathbb{R}^2)} + \|F_N\|_{L^p(\mathbb{R}^2)}$$

Using $\|F_n\|_{L^p(\mathbb{R}^2)} \lesssim_{\chi_0} |a_n| \delta^6 N^{\frac{6}{p}}$ and $D_p^{(2)}(\delta) \geq 1$, we simplify the upper bound:

$$\|F\|_{L^p(\mathbb{R}^2)} \lesssim_{\chi_0} D_p^{(2)}(\delta) \delta^6 N^{\frac{6}{p}} \left(\sum_{n=0}^{N} |a_n|^2 \right)^{\frac{1}{2}}$$

Combining this with the lower bound (canceling $\delta^6 N^{\frac{6}{p}}$):

$$\left(\int_0^1 \int_0^1 \left| \sum_{n=0}^{N} a_n e^{2\pi i \left(nx + n^3 t\right)} \right|^p dxdt \right)^{\frac{1}{p}} \lesssim_{\chi_0} D_p^{(2)}(\delta) \left(\sum_{n=0}^{N} |a_n|^2 \right)^{\frac{1}{2}}$$

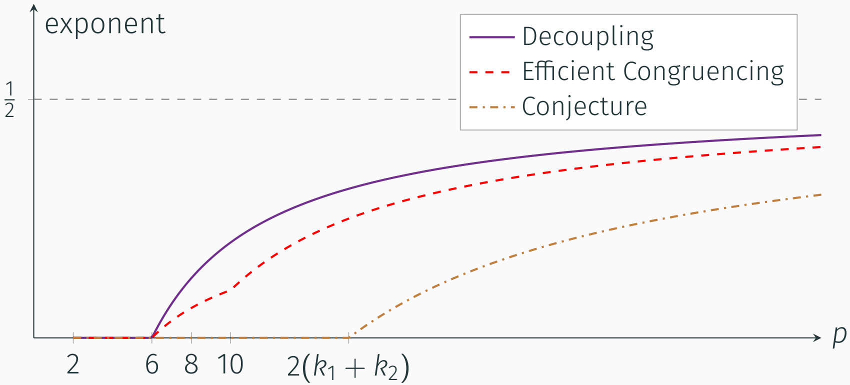
Substituting $D_p^{(2)}(\delta) \lesssim_{p,\varepsilon} N^{\varepsilon} \left(1 + N^{\frac{1}{2} - \frac{3}{p}}\right)$ finishes the proof.

Discrete Strichartz Estimates for $(\mathrm{Poly}_{k_1}(n), \mathrm{Poly}_{k_2}(n))$

Theorem (Wooley, 2022)

Let $\deg P = k_1$ and $\deg Q = k_2$ satisfying P' and Q' are linearly independent over $\mathbb{Q}$ and $\max\{k_1, k_2\} \geq 3$, then for $p \in [2, 6] \cup [10, \infty)$,

$$\left\| \sum_{n=1}^{N} a_n e^{2\pi i (P(n)x + Q(n)t)} \right\|_{L^p([0,1]^2)} \lesssim_{P,Q,p,\varepsilon} N^{\varepsilon} \left(1 + N^{\frac{1}{2} - \frac{4}{p}}\right) \left(\sum_{n=1}^{N} |a_n|^2\right)^{\frac{1}{2}}.$$

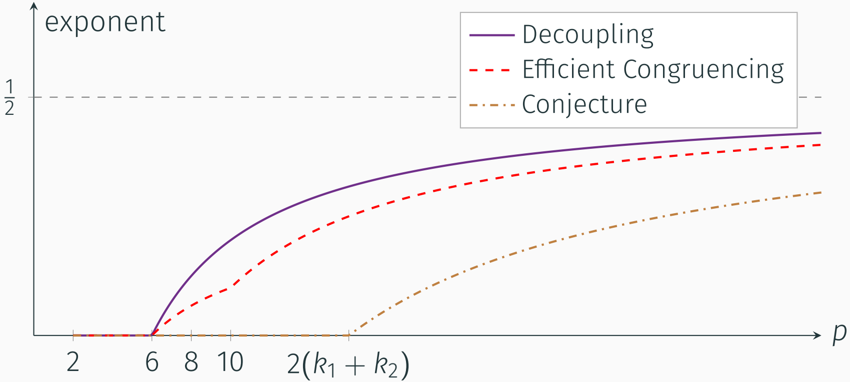


Discrete Strichartz Estimates for $(\mathrm{Poly}_{k_1}(n), \mathrm{Poly}_{k_2}(n))$

Conclusion by Decoupling (C., 2026+)

Let $\deg P = k_1$ and $\deg Q = k_2$ satisfying P' and Q' are linearly independent over $\mathbb{Q}$, then

$$\left\| \sum_{n=1}^{N} a_n e^{2\pi i (P(n)x + Q(n)t)} \right\|_{L^p([0,1]^2)} \lesssim_{P,Q,p,\varepsilon} N^{\varepsilon} \left(1 + N^{\frac{1}{2} - \frac{3}{p}}\right) \left(\sum_{n=1}^{N} |a_n|^2 \right)^{\frac{1}{2}}.$$



Sketch of the Proof for $(\mathrm{Poly}_{k_1}(n), \mathrm{Poly}_{k_2}(n))$

It suffices to prove the endpoint estimate

$$\left\| \sum_{n=1}^{N} a_n e^{2\pi i (P(n)x + Q(n)t)} \right\|_{L^6([0,1]^2)} \lesssim_{P,Q,\varepsilon} N^{\varepsilon} \left(\sum_{n=1}^{N} |a_n|^2 \right)^{1/2} .$$

If $\mathbf{deg}\, P = \mathbf{deg}\, Q$, replace Q by $R = AQ - BP$, where A, B are the leading coefficients of P, Q. Then $\mathbf{deg}\, R < \mathbf{deg}\, P$, and a fixed linear change of variables reduces the problem to the case

$$1 \le k_1 = \mathbf{deg}\, P < k_2 = \mathbf{deg}\, Q.$$

On a dyadic block $n \sim M$, consider the rescaled curve

$$\Psi_M(u) = \left(\frac{P(Mu)}{M^{k_1}}, \frac{Q(Mu)}{M^{k_2}} \right), \qquad u \in [1/2, 1].$$

As $M \to \infty$, this converges in C^2 to a non-degenerate monomial curve. Therefore we use the non-degenerate curve decoupling inequality:

$$\|F\|_{L^6(\mathbb{R}^2)} \lesssim_{P,Q,\varepsilon} M^{\varepsilon} \left(\sum_{J \in \mathcal{I}_{M-1}} \|F_J\|_{L^6(\mathbb{R}^2)}^2 \right)^{1/2} ,$$

for functions whose Fourier support lies in an M^{-2}neighborhood of Ψ_M.

Sketch of the Proof for $(\mathrm{Poly}_{k_1}(n), \mathrm{Poly}_{k_2}(n))$

Define the continuous extension on the dyadic block by

$$\widehat{F}_M(\xi_1, \xi_2) = \sum_{n \sim M} a_n \chi_0 \left(A_M \left(\xi_1 - \frac{P(n)}{M^{k_1}} \right), A_M \left(\xi_2 - \frac{Q(n)}{M^{k_2}} \right) \right), \qquad A_M = cM^{k_2}.$$

Since $A_M^{-1} \lesssim M^{-2}$, its Fourier support lies in the admissible decoupling neighborhood.

The decoupling inequality gives

$$\|F_M\|_{L^6(\mathbb{R}^2)} \lesssim_{P,Q,\varepsilon} M^{\varepsilon} A_M^{-2} M^{k_2/3} \left(\sum_{n \sim M} |a_n|^2 \right)^{1/2}.$$

On the other hand, after the change of variables $x = M^{k_1} y_1, t = M^{k_2} y_2$, and using the integer periodicity of $P(n)$ and $Q(n)$,

$$\|F_M\|_{L^6(\mathbb{R}^2)} \gtrsim A_M^{-2} M^{k_2/3} \left\| \sum_{n \sim M} a_n e^{2\pi i (P(n) y_1 + Q(n) y_2)} \right\|_{L^6([0,1]^2)}.$$

The common factor cancels, giving the L^6 estimate on each dyadic block. Summing over blocks and interpolating with L^2 and L^∞ yields

$$\left\| \sum_{n=1}^{N} a_n e^{2\pi i (P(n)x + Q(n)t)} \right\|_{L^p([0,1]^2)} \lesssim N^{\varepsilon} \left(1 + N^{1/2 - 3/p} \right) \left(\sum_{n=1}^{N} |a_n|^2 \right)^{1/2}.$$

In the two examples above, the decoupling method did not yield very good results.

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But in the following examples, the decoupling method will really stand out.

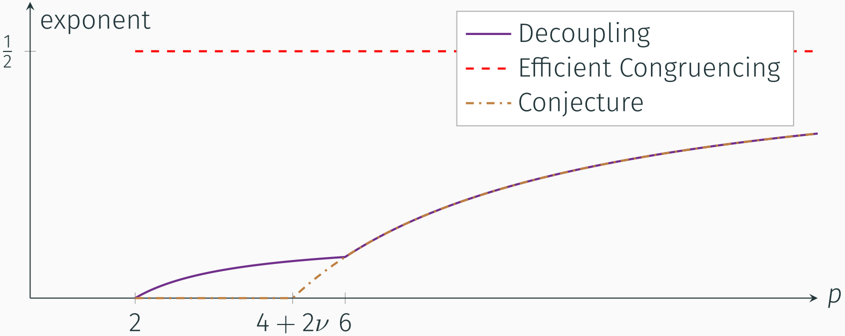
Discrete Strichartz Estimates for $(n, n^{1+\nu})$

Theorem (C., 2026+)

Let $\nu > 0$, then

$$\left\| \sum_{n=1}^{N} a_n e^{2\pi i(nx+n^{1+\nu}t)} \right\|_{L^p([0,1]^2)} \lesssim_{\nu,p,\varepsilon} N^{E_p+\varepsilon} \left(\sum_{n=1}^{N} |a_n|^2 \right)^{\frac{1}{2}},$$

where $E_p = \begin{cases} \frac{(r-1-\nu)(p-2)}{4p}, & \text{if } 2 \le p \le 6, \\ \frac{1}{2} + \frac{r-4-\nu}{p}, & \text{if } p > 6. \end{cases}$ is defined with $r = \max\{2, 1+\nu\}$.



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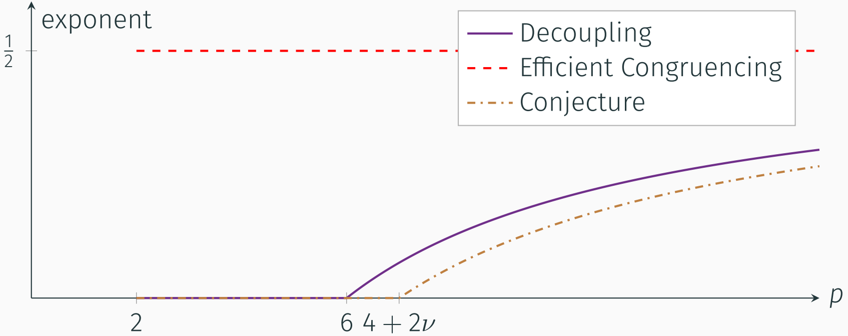

Sketch of the Proof for $(n, n^{1+\nu})$

Since we can use interpolation, it suffices to prove the endpoint $p = 6$.

Let $r = \max\{2, 1+\nu\}$. The required decoupling input is the refined local decoupling inequality, which is mentioned before, for $\gamma_\nu(s) = (s, s^{1+\nu})$.

For

$$\Omega_I^{(\nu)}(\delta) = \{(\xi_1, \xi_1^{1+\nu} + \eta) : \xi_1 \in I,\ |\eta| \leq \delta^r\},$$

for a rectangle R centered at x_R with side lengths $L_1 \times L_2$, define

$$\omega_R(x) = \left(1 + \left|\left(\frac{x_1 - x_{R,1}}{L_1}, \frac{x_2 - x_{R,2}}{L_2}\right)\right|\right)^{-200}.$$

one has, for $2 \leq p \leq 6$,

$$\|F\|_{L^p(\omega_{R_{\delta,r}})} \lesssim_{\nu,p,\varepsilon} \delta^{-\varepsilon} \left(\sum_{I \in \mathcal{I}_\delta} \|F_I\|^2_{L^p(\omega_{R_{\delta,r}})}\right)^{1/2},$$

where $\|F\|_{L^p(\omega_R)} = \left(\int_{\mathbb{R}^d} |F(x)|^p \, \omega_R(x) \, dx\right)^{1/p}$, whenever $\operatorname{supp} \widehat{F} \subset \Omega^{(\nu)}_{[0,1]}(\delta)$, where $R_{\delta,r}$ has side lengths $\delta^{-2} \times \delta^{-r}$.

Sketch of the Proof for $(n, n^{1+\nu})$

Discard the harmless endpoint $n = N$. Define

$$\widehat{F}(\xi_1, \xi_2) = \sum_{n=1}^{N-1} a_n \chi_0 \left(N^r \left(\xi_1 - \frac{n}{N} \right), N^r \left(\xi_2 - \left(\frac{n}{N} \right)^{1+\nu} \right) \right).$$

The support lies inside the admissible neighborhood of $(s, s^{1+\nu})$ at scale $\delta = N^{-1}$. Let $R_{N^{-1},r}$ have side lengths $N^2 \times N^r$, centered at the midpoint of $[0, N^2] \times [0, N^{1+\nu}]$. Decoupling gives

$$\|F\|_{L^6(\omega_{R_{N^{-1},r}})} \lesssim_{\nu,\varepsilon} N^{-2r+(r+2)/6+\varepsilon} \left(\sum_{n=1}^{N-1} |a_n|^2 \right)^{1/2}.$$

On the rectangle $[0, N^2] \times [0, N^{1+\nu}]$, the inverse Fourier cutoff and the weight are bounded below. With $y_1 = Nx, y_2 = N^{1+\nu}t$, and using periodicity in x, we obtain

$$\|F\|_{L^6(\omega_{R_{N^{-1},r}})} \gtrsim N^{-2r+(3+\nu)/6} \left\| \sum_{n=1}^{N-1} a_n e^{2\pi i (nx + n^{1+\nu} t)} \right\|_{L^6([0,1]^2)}.$$

Combining the two bounds gives

$$\left\| \sum_{n=1}^{N} a_n e^{2\pi i (nx + n^{1+\nu} t)} \right\|_{L^6([0,1]^2)} \lesssim N^{(r-1-\nu)/6+\varepsilon} \left(\sum_{n=1}^{N} |a_n|^2 \right)^{1/2}.$$

Two Common Questions

Why there is a polynomial loss?

Decoupling gives control on a larger box with size of $N^2 \times N^r$, which is the upper bound. For the lower bound, we can only use periodicity to get control for a box with size of $N^2 \times N^{1+\nu}$. Then $N^{(r-1-\nu)/6}$ is obtained by dividing one by the other and taking the 6th root.

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Why we need $r = \max\{1+\nu, 2\}$?

The curve is still $(s, s^{1+\nu})$, but the decoupling scale cannot be better than the parabolic scale 2, since we consider that the second order derivative (which is $s^{\nu-1}$) will blow up otherwise. If the curve is flatter than a parabola we use its finite type $1+\nu$, otherwise the baseline scale is 2.

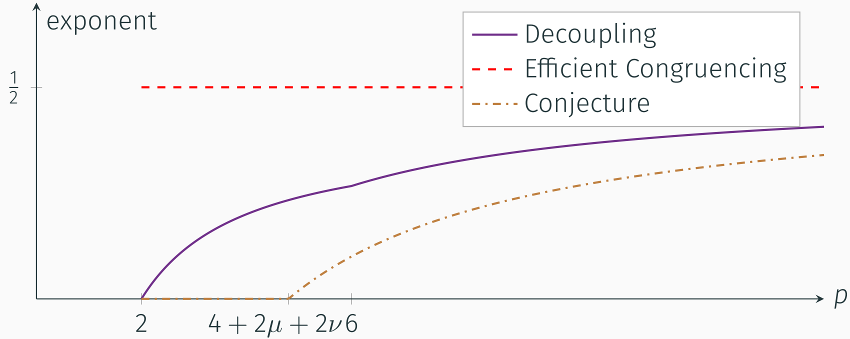
Discrete Strichartz Estimates for $(n^{1+\mu}, n^{1+\nu})$

Theorem (C., 2026+)

Let $0 < \mu < \nu$, then

$$\left\| \sum_{n=1}^{N} a_n e^{2\pi i (x n^{1+\mu} + t n^{1+\nu})} \right\|_{L^p([0,1]^2)} \lesssim_{\mu,\nu,p,\varepsilon} N^{E_p+\varepsilon} \left(\sum_{n=1}^{N} |a_n|^2 \right)^{\frac{1}{2}},$$

where $E_p = \begin{cases} \frac{(p-2)(r_1+r_2-2-\mu-\nu)}{4p}, & 2 \le p \le 6, \\ \frac{1}{2} + \frac{r_1+r_2-5-\mu-\nu}{p}, & p > 6. \end{cases}$ is defined with $r_1 = \max\{2, 1+\mu\}, r_2 = \max\{2, 1+\nu\}$.



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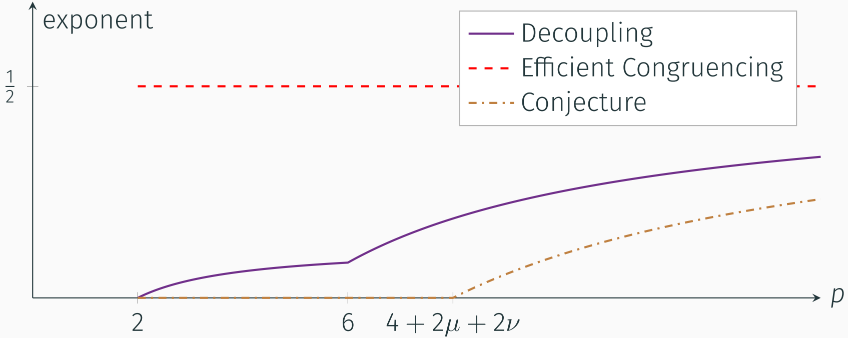

Discrete Strichartz Estimates for $(n^{1+\mu}, n^{1+\nu})$

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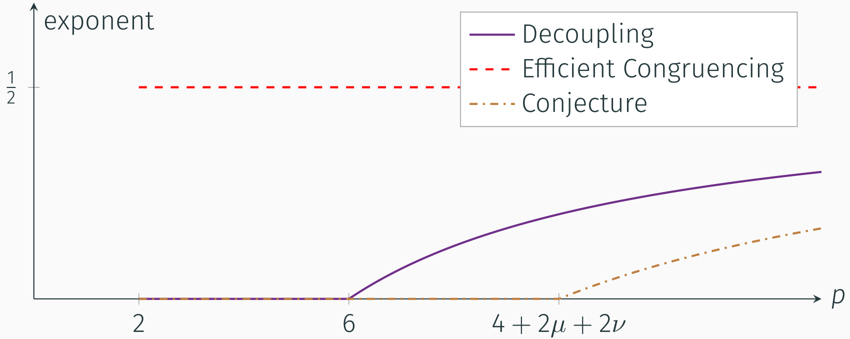


Sketch of the Proof for $(n^{1+\mu}, n^{1+\nu})$

We only prove the case of $p = 6$. Let $r_1 = \max\{2, 1+\mu\}$, $r_2 = \max\{2, 1+\nu\}$. For the curve $\Gamma(s) = (s^{1+\mu}, s^{1+\nu})$, use the anisotropic neighborhood

$$\Omega_I^\Gamma(\delta) = \{(s^{1+\mu} + \eta_1, s^{1+\nu} + \eta_2) : s \in I,\ |\eta_1| \le \delta^{r_1},\ |\eta_2| \le \delta^{r_2}\}.$$

The needed decoupling inequality is

$$\|F\|_{L^p(\omega_{R_\delta})} \lesssim_{\mu,\nu,p,\varepsilon} \delta^{-\varepsilon} \left(\sum_{J \in \mathcal{I}_\delta} \|F_J\|^2_{L^p(\omega_{R_\delta})} \right)^{1/2}, \qquad 2 \le p \le 6,$$

where R_δ has side lengths $\delta^{-r_1} \times \delta^{-r_2}$. The proof of this decoupling inequality follows the same dyadic rescaling principle as in the finite-type discussion: near $s = 0$, isolate a small core; on each dyadic block $s \sim a$, write $s = au$. The curve becomes non-degenerate on $u \in [1, 2]$, and the rescaled frequency widths satisfy

$$\delta^{r_1} a^{-(1+\mu)} \le (\delta/a)^2, \qquad \delta^{r_2} a^{-(1+\nu)} \le (\delta/a)^2.$$

Thus ordinary non-degenerate curve decoupling applies on each block.

Sketch of the Proof for $(n^{1+\mu}, n^{1+\nu})$

Discard the harmless endpoint $n = N$. Define

$$\widehat{F}(\xi_1,\xi_2) = \sum_{n=1}^{N-1} a_n \chi_0 \left(N^{r_1}\left(\xi_1 - \left(\frac{n}{N}\right)^{1+\mu}\right), N^{r_2}\left(\xi_2 - \left(\frac{n}{N}\right)^{1+\nu}\right)\right).$$

Choose R_{N-1} with side lengths $N^{r_1} \times N^{r_2}$, centered at the midpoint of $[0, N^{1+\mu}] \times [0, N^{1+\nu}]$. The anisotropic decoupling inequality gives

$$\|F\|_{L^6(\omega_{R_{N-1}})} \lesssim_{\mu,\nu,\varepsilon} N^{\varepsilon} N^{-5(r_1+r_2)/6} \left(\sum_{n=1}^{N-1} |a_n|^2\right)^{1/2}.$$

On the rectangle $[0, N^{1+\mu}] \times [0, N^{1+\nu}]$, the cutoff and the weight are bounded below. With $y_1 = N^{1+\mu}x, y_2 = N^{1+\nu}t$, we obtain

$$\|F\|_{L^6(\omega_{R_{N-1}})} \gtrsim N^{-r_1-r_2+(2+\mu+\nu)/6} \left\|\sum_{n=1}^{N-1} a_n e^{2\pi i(xn^{1+\mu}+tn^{1+\nu})}\right\|_{L^6([0,1]^2)}.$$

Therefore

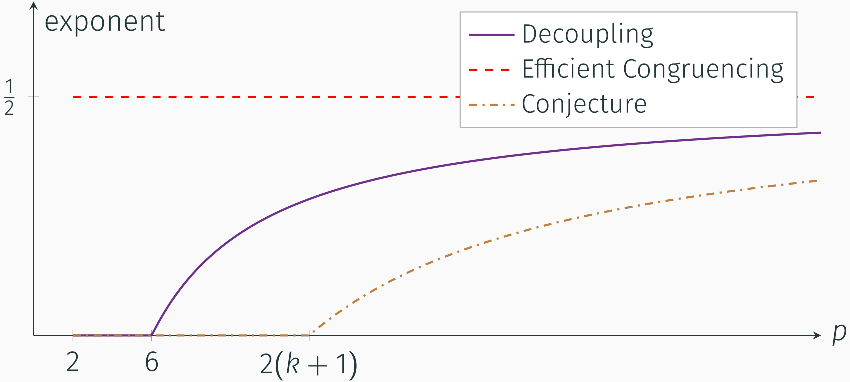
$$\left\|\sum_{n=1}^{N} a_n e^{2\pi i(xn^{1+\mu}+tn^{1+\nu})}\right\|_{L^6([0,1]^2)} \lesssim N^{(r_1+r_2-2-\mu-\nu)/6+\varepsilon} \left(\sum_{n=1}^{N} |a_n|^2\right)^{1/2}.$$

Discrete Strichartz Estimates for $(n, n^k + O(1))$

Theorem (C., 2026+)

Let integer $k \geq 2$, real sequence $\{\varepsilon_n\}_{n=0}^N$ satisfying $\sup_{0\leq n\leq N} |\varepsilon_n| \leq C$, then

$$\left\| \sum_{n=0}^{N} a_n e^{2\pi i (nx + (n^k + \varepsilon_n)t)} \right\|_{L^p([0,1]^2)} \lesssim_{k,p,\varepsilon,C} N^{\varepsilon} \left(1 + N^{\frac{1}{2} - \frac{3}{p}}\right) \left(\sum_{n=0}^{N} |a_n|^2 \right)^{\frac{1}{2}}.$$



Sketch of the Proof for $(n, n^k + O(1))$

This conclusion is similar to the case of $(n, n^{1+\nu})$, with $\nu = k-1, r = k$. We use the local weighted decoupling inequality for $\gamma(s) = (s, s^k)$.

At scale $\delta = N^{-1}$, the relevant tube is $\Omega^{(k-1)}_{[0,1]}(N^{-1}) = \{(\xi_1, \xi_1^k + \eta) : 0 \le \xi_1 \le 1,\ |\eta| \lesssim N^{-k}\}$, and for $p = 6$, with $R_{N^{-1},k}$ of side lengths $N^2 \times N^k$,

$$\|F\|_{L^6(\omega_{R_{N^{-1},k}})} \lesssim_{k,\varepsilon} N^{\varepsilon} \left(\sum_{I \in \mathcal{I}_{N^{-1}}} \|F_I\|^2_{L^6(\omega_{R_{N^{-1},k}})} \right)^{1/2}.$$

The only new point is that the perturbed frequencies still lie in the same tube. Indeed, for $1 \le n \le N-1$, the frequency center is $\left(\frac{n}{N}, \frac{n^k+\varepsilon_n}{N^k}\right)$, and its vertical distance from (s, s^k) at $s = n/N$ is $\left|\frac{n^k+\varepsilon_n}{N^k} - \left(\frac{n}{N}\right)^k\right| = \frac{|\varepsilon_n|}{N^k} \le CN^{-k}$.

The same frequency-bump construction gives

$$\|F\|_{L^6(\omega_{R_{N^{-1},k}})} \lesssim N^{-2k+(k+2)/6+\varepsilon} \left(\sum_{n=1}^{N-1} |a_n|^2 \right)^{1/2}.$$

On $[0, N^2] \times [0, N^k]$, the cutoff and the weight are bounded below. With $y_1 = Nx,\ y_2 = N^k t$, periodicity in x gives the matching lower bound, so the common factor cancels and we get

$$\left\| \sum_{n=0}^{N} a_n e^{2\pi i (nx + (n^k+\varepsilon_n)t)} \right\|_{L^p([0,1]^2)} \lesssim_{k,p,\varepsilon,C} N^{\varepsilon} \left(1 + N^{1/2-3/p}\right) \left(\sum_{n=0}^{N} |a_n|^2 \right)^{1/2}.$$

Example

As an example, we can take $\varepsilon_n = \mathbf{1}_{\mathbb{P}}(n)$ as the characteristic function of the prime numbers. Then the last theorem gives

$$\left\| \sum_{n=0}^{N} a_n e^{2\pi i (nx + (n^k + \mathbf{1}_{\mathbb{P}}(n))t)} \right\|_{L^p([0,1]^2)} \lesssim_{k,p,\varepsilon} N^{\varepsilon} \left(1 + N^{\frac{1}{2} - \frac{3}{p}}\right) \left(\sum_{n=0}^{N} |a_n|^2 \right)^{\frac{1}{2}},$$

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(If we really want to use efficient congruencing methods, we need to study the number of solutions for

$$x_1 + \cdots + x_s = x_{s+1} + \cdots + x_{2s}$$
$$(x_1^k + \mathbf{1}_{\mathbb{P}}(x_1)) + \cdots + (x_s^k + \mathbf{1}_{\mathbb{P}}(x_s)) = (x_{s+1}^k + \mathbf{1}_{\mathbb{P}}(x_{s+1})) + \cdots + (x_{2s}^k + \mathbf{1}_{\mathbb{P}}(x_{2s}))$$

which is hard to predict.)

Summary for $\left\|\sum_{n=1}^{N} a_n e^{2\pi i(\boxed{g(n)}x+\boxed{h(n)}t)}\right\|_{L^p([0,1]^2)} \lesssim_{p,\varepsilon} N^{\boxed{E_p}+\varepsilon}\left(\sum_{n=1}^{N} |a_n|^2\right)^{\frac{1}{2}}$

$(g(n),h(n))$	E_p by decoupling methods	E_p by efficient congruencing methods	Conjecture E_p
(n,n^2)	Bourgain-Demeter15 $\left(\frac{1}{2}-\frac{3}{p}\right)_+$	Wooley19 $\left(\frac{1}{2}-\frac{3}{p}\right)_+$	$\left(\frac{1}{2}-\frac{3}{p}\right)_+$
(n,n^3)	C.26+ $\left(\frac{1}{2}-\frac{3}{p}\right)_+$	Wooley22 $\left(\frac{1}{2}-\frac{4}{p}\right)_+$ for $p\in(2,6]\cup[10,\infty)$	$\left(\frac{1}{2}-\frac{4}{p}\right)_+$
$\begin{matrix}(Poly_{k_1}(n),Poly_{k_2}(n))\\ 1\le k_1\le k_2\end{matrix}$	C.26+ $\left(\frac{1}{2}-\frac{3}{p}\right)_+$	Wooley22 $\left(\frac{1}{2}-\frac{4}{p}\right)_+$ for $p\in(2,6]\cup[10,\infty)$	$\left(\frac{1}{2}-\frac{k_1+k_2}{p}\right)_+$
$(n,n^{1+\nu})$	C.26+ $\begin{cases}\frac{(r-1-\nu)(p-2)}{4p}, & 2\le p\le 6,\\ \frac{1}{2}+\frac{r-4-\nu}{p}, & p>6.\end{cases}$ where $r=\max\{2,1+\nu\}$	(trivial bound) $\frac{1}{2}$	$\left(\frac{1}{2}-\frac{2+\nu}{p}\right)_+$
$\begin{matrix}(n^{1+\mu},n^{1+\nu})\\ 0<\mu<\nu\end{matrix}$	C.26+ $\begin{cases}\frac{(p-2)(r_1+r_2-2-\mu-\nu)}{4p}, & 2\le p\le 6,\\ \frac{1}{2}+\frac{r_1+r_2-5-\mu-\nu}{p}, & p>6.\end{cases}$ where $r_1=\max\{2,1+\mu\}, r_2=\max\{2,1+\nu\}$	(trivial bound) $\frac{1}{2}$	$\left(\frac{1}{2}-\frac{2+\mu+\nu}{p}\right)_+$
$(n,n^k+O(1))$	C.26+ $\left(\frac{1}{2}-\frac{3}{p}\right)_+$	(trivial bound) $\frac{1}{2}$	$\left(\frac{1}{2}-\frac{k+1}{p}\right)_+$

Here we uses notations $(\cdot)_+$ as a short of $\max\{0,\cdot\}$ and $Poly_k$ as a polynomial with integer coefficients and degree of k. Cells with gray background indicate that the bound obtained by the corresponding method is currently the best.

Final Remark

However, the decoupling method also suffers from a rather evident drawback: specifically (in the two-dimensional setting), it yields sharp bounds only for cases where $p \leq 6$. Yet, in certain scenarios, such as KdV version of discrete Strichartz, we aim to establish sharp bounds for cases extending up to $p \leq 8$; in these situations, the efficient congruencing method proves to be the more effective tool.

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Thank you!

Any Questions?

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