Polynomial Partitioning and Its Applications in Incidence Geometry

SHAC Seminar

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Nankai/UC Berkeley

Incidences and Szemerédi-Trotter Theorem

Definition (Incidence)

If $\mathcal{S}$ is a set of points and $\mathfrak{L}$ is a set of lines (or curves), the set of incidences is defined as $I(\mathcal{S}, \mathfrak{L}) = \{(p, \ell) \in \mathcal{S} \times \mathfrak{L} \mid p \in \ell\}$.

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Theorem (Szemerédi-Trotter)

If $\mathcal{S}$ is a set of S points in the plane, and $\mathfrak{L}$ is a set of L lines in the plane, then the number of incidences is bounded by:

$$|I(\mathcal{S}, \mathfrak{L})| \lesssim \left(S^{2/3}L^{2/3} + S + L\right)$$

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There are many ways to prove this theorem, such as crossing number method, cutting method, and polynomial partitioning method (today's topic).

The cutting method is a divide-and-conquer approach: cut the plane into pieces, estimate incidences in each piece, and add up contributions.

A Rough Estimate

Theorem (Kővári-Sós-Turán)

If an $M \times N$ 0-1 matrix contains no 2×2 all-ones submatrix, then the number of 1's is at most $M\sqrt{N} + N$.

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Let $\mathcal{S}$ be a set of $|\mathcal{S}| = S$ points and $\mathfrak{L}$ a set of $|\mathfrak{L}| = L$ lines. Then

$$|I(\mathcal{S}, \mathfrak{L})| \leq S\sqrt{L} + L \qquad \text{and} \qquad |I(\mathcal{S}, \mathfrak{L})| \leq L\sqrt{S} + S.$$

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Proof sketch.

The incidence matrix is $S \times L$ with entries 1 if point lies on line, else 0. Two distinct lines intersect in at most one point, and two distinct points determine at most one line, hence the matrix contains no 2×2 all-ones minor. Applying Kővári-Sós-Turán's Theorem (with $M = S$, $N = L$ and its transpose) gives the two bounds. □

Cells and Equidistribution

We use D auxiliary lines to cut the plane into $\sim D^2$ cells.

Lemma

A line can enter at most $D + 1$ of the cells determined by D lines.

Proof.

To go from one cell to another, a line must cross one of the D auxiliary lines. But a given line intersects each of the D auxiliary lines at most once. $\square$

We ideally want points and lines to be *equidistributed*:

$$\text{Each cell intersects } \lesssim LD^{-1} \text{ lines of } \mathfrak{L} \quad (EquiL)$$

$$\text{Each cell contains } \lesssim SD^{-2} \text{ points of } \mathcal{S} \quad (EquiS)$$

If (EquiL) and (EquiS) hold, applying the rough estimate (last page) for every cell, then for every cell we have $|I(\mathcal{S}_{single}, \mathcal{L}_{single})| \lesssim SL^{1/2}D^{-5/2} + LD^{-1}$. So we get $|I(\mathcal{S}, \mathfrak{L})| \lesssim D^{-1/2}SL^{1/2} + DL$. Optimizing D implies Szemerédi-Trotter bound.

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Counterexample: Let $\mathcal{S}$ lie on a strictly convex closed curve γ (e.g., a circle). D auxiliary lines cut γ into at most $2D$ pieces, so points lie in at most $2D$ cells instead of $\sim D^2$. One cell will have $\gtrsim SD^{-1}$ points (much more than SD^{-2}).

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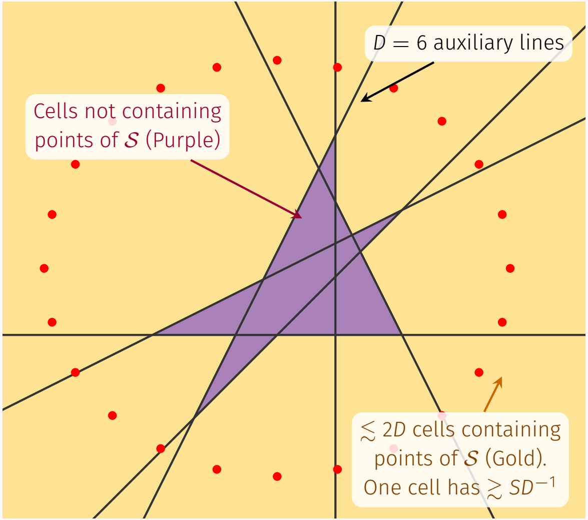

$D = 6$ auxiliary lines

Cells not containing points of $\mathcal{S}$ (Purple)

$\lesssim 2D$ cells containing points of $\mathcal{S}$ (Gold). One cell has $\gtrsim SD^{-1}$

Polynomial Partitioning Theorem

Theorem (Oleinik-Petrovski, Milnor, and Thom)

For any dimension n, we can choose $C(n)$ so that the following holds. If X is any finite subset of $\mathbb{R}^n$ and D is any degree, then there is a non-zero polynomial $P \in \mathrm{Poly}_D(\mathbb{R}^n)$ so that $\mathbb{R}^n \backslash Z(P)$ is a disjoint union of $\lesssim D^n$ open sets O_i each containing $\leq C(n)|X|D^{-n}$ points of X.

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"The proof isn't important; the examples are what really matter."

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Caveat: The theorem does NOT guarantee that the points of X lie in the complement of $Z(P)$. We must handle $X \cap Z(P)$ separately.

Polynomial Partitioning Theorem

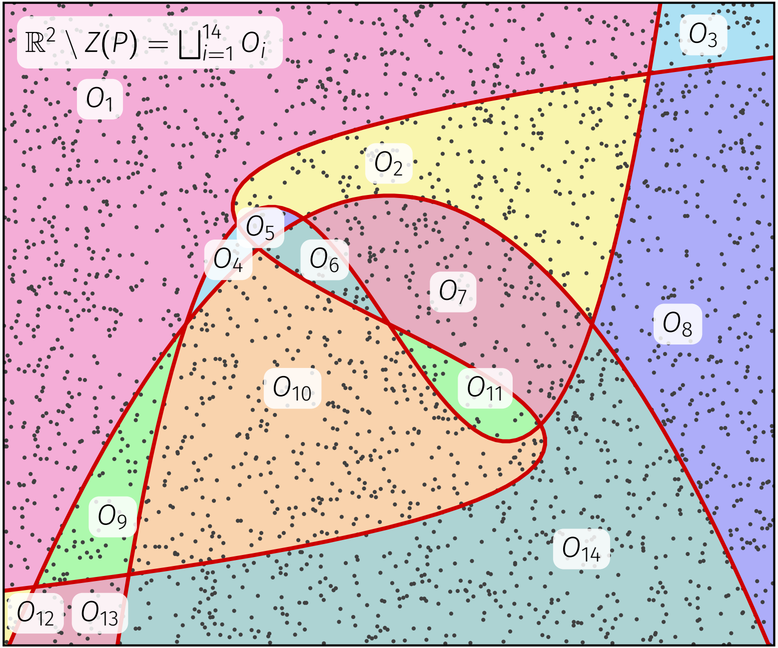

Illustration of Polynomial Partitioning Theorem: Randomized Polynomial Cell Decomposition in $\mathbb{R}^2$ (Degree $D = 8$)

Point Count Equality: $|X \cap O_i| \leq C(n)\frac{|X|}{D^n}$ (here $|X| = 2500$, each O_i contains $\leq C(n)\frac{2500}{64}$ points)

● are points in $X \subset \mathbb{R}^2$

Red Curves are $Z(P)$

Reproving Szemerédi-Trotter Theorem—Step 1: Counting Lemma

Counting lemma

We have $|I(\mathcal{S}, \mathfrak{L})| \leq L + S^2 \quad \text{and} \quad |I(\mathcal{S}, \mathfrak{L})| \leq L^2 + S$.

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Proof.

Fix $x \in \mathcal{S}$. Let L_x be the number of lines of $\mathfrak{L}$ that contain x and no other point of $\mathcal{S}$. For each other point $y \in \mathcal{S}$, there is at most one line of $\mathfrak{L}$ containing x and y.
Therefore, $|I(x, \mathfrak{L})| \leq S + L_x$. So $|I(\mathcal{S}, \mathfrak{L})| \leq S^2 + \sum_{x \in S} L_x \leq S^2 + L$.
The proof of the other inequality is similar. ☐

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The proof of the other inequality is similar. □

If $L > S^2$ or $S > L^2$, the counting lemma suffices. We can restrict to $S^{1/2} \leq L \leq S^2$.

Reproving Szemerédi-Trotter Theorem—Step 2: Cell Incidences

Choose $P \in \mathrm{Poly}_D(\mathbb{R}^2)$ partitioning $\mathcal{S}$ into $\lesssim D^2$ cells O_i, each with $S_i \lesssim SD^{-2}$ points, by the polynomial partitioning theorem. For each i, let $\mathcal{S}_i = \mathcal{S} \cap O_i$ and let $\mathfrak{L}_i \subset \mathfrak{L}$ be the set of lines of $\mathfrak{L}$ that intersect O_i. Let $S_i = |\mathcal{S}_i|$ and $L_i = |\mathfrak{L}_i|$.

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If a line $\notin Z(P)$, then it intersects $Z(P)$ in at most D points, and so each line intersects at most $D + 1$ cells. Therefore, $\sum_i L_i \leq (D + 1)L$.

Applying counting lemma in each cell, we get $|I(\mathcal{S}_i, \mathfrak{L}_i)| \leq L_i + S_i^2$.

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Let $\mathcal{S}_{\text{cell}}$ be the union of $\mathcal{S}_i$ — all the points of $\mathcal{S}$ that lie in the interiors of the cells.

$$|I(\mathcal{S}_{\text{cell}}, \mathfrak{L})| = \sum_i |I(\mathcal{S}_i, \mathfrak{L}_i)| \leq \sum_i L_i + \sum_i S_i^2 \lesssim LD + SD^{-2} \sum_i S_i \leq LD + S^2 D^{-2}.$$

Reproving Szemerédi-Trotter Theorem—Step 3: Algebraic Incidences

We let $\mathcal{S}_{\text{alg}}$ be the set of points in $Z(P)$, which implies $\mathcal{S} = \mathcal{S}_{\text{cell}} \cup \mathcal{S}_{\text{alg}}$. It remains to bound $|I\left(\mathcal{S}_{\text{alg}}, \mathfrak{L}\right)|$. We divide $\mathfrak{L}$ as $\mathfrak{L}_{\text{alg}} \cup \mathfrak{L}_{\text{cell}}$, where $\mathfrak{L}_{\text{alg}}$ are the lines contained in $Z(P)$ and $\mathfrak{L}_{\text{cell}}$ are the other lines. The total number of incidences is bounded by

$$|I(\mathcal{S}, \mathcal{L})| \leq |I\left(\mathcal{S}_{\text{cell}}, \mathcal{L}\right)| + \left|I\left(\mathcal{S}_{\text{alg}}, \mathcal{L}_{\text{cell}}\right)\right| + \left|I\left(\mathcal{S}_{\text{alg}}, \mathcal{L}_{\text{alg}}\right)\right|.$$

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Each line of $\mathfrak{L}_{\text{cell}}$ has at most D intersection points with $Z(P)$, and so it has at most D incidences with $\mathcal{S}_{\text{alg}}$. Therefore $\left|I\left(\mathcal{S}_{\text{alg}}, \mathfrak{L}_{\text{cell}}\right)\right| \leq LD$.

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The number of lines in $\mathfrak{L}_{\text{alg}}$ is at most D. By counting lemma,

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Combined with $|I(\mathcal{S}_{\text{cell}}, \mathfrak{L})| \lesssim LD + S^2 D^{-2}$, we get the total incidences:

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Optimizing with $D \sim S^{2/3}L^{-1/3}$ yields $|I(\mathcal{S}, \mathfrak{L})| \lesssim S^{2/3}L^{2/3} + S$, where we need to use the assumption $S^{1/2} \leq L \leq S^2$. Now the proof is finished.

Point-Circle Incidences

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Point-Circle Incidences Estimate

Suppose that Γ is a set of N circles in the plane and $\mathcal{S}$ is a set of S points in the plane. We have

$$|I(\mathcal{S},\Gamma)| \leq S^{3/5}N^{4/5} + N + S.$$

Point-Circle Incidences—Step 1: Counting Lemma

Counting Lemma (Point-Circle)

If $\mathcal{S}$ is a set of S points in $\mathbb{R}^2$, and Γ is a set of N circles in $\mathbb{R}^2$, then
(1) $|I(\mathcal{S}, \Gamma)| \leq 2N + S^3$;
(2) $|I(\mathcal{S}, \Gamma)| \leq 2N^2 + S$.

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Proof.

(1) Consider the circumcenter of a triangle, we can know a set of 3 points lies on a unique circle immediately. Fix $x \in \mathcal{S}$. Let $\Gamma_{high} \subseteq \Gamma$ be the circles containing at least 3 points in $\mathcal{S}$ and $\Gamma_{low} = \Gamma \backslash \Gamma_{high}$. Therefore,
$|I(x,\Gamma)| = |I(x,\Gamma_{high})| + |I(x,\Gamma_{low})| \leq \binom{S-1}{2} + |I(x,\Gamma_{low})|$. So
$|I(\mathcal{S},\Gamma)| \leq S \cdot S^2 + \sum_{x\in\mathcal{S}} |I(x,\Gamma_{low})| \leq S^3 + 2N$.

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$|I(\mathcal{S},\Gamma)| \leq S \cdot S^2 + \sum_{x\in\mathcal{S}} |I(x,\Gamma_{low})| \leq S^3 + 2N$.
(2) Fix $\gamma \in \Gamma$. Let S_γ be the number of points of $\mathcal{S}$ that lie on γ and no other circle of Γ. For each other circle $\delta \in \Gamma$, there are at most two points of $\mathcal{S}$ contained in both γ and δ. Therefore, $|I(\mathcal{S},\gamma)| \leq 2N + S_\gamma$. So
$|I(\mathcal{S},\Gamma)| \leq 2N^2 + \sum_{\gamma\in\Gamma} S_\gamma \leq 2N^2 + S$. □

Point-Circle Incidences—Step 2: Cell Incidences

We can find a polynomial $P \in \mathrm{Poly}_D$ such that each cell of $\mathbb{R}^2 \setminus Z(P)$ contains $\lesssim SD^{-2}$ points of $\mathcal{S}$. Let O_i be these cells, $\mathcal{S}_i = \mathcal{S} \cap O_i$, and Γ_i the circles intersecting O_i. Write $S_i = |\mathcal{S}_i|$, $N_i = |\Gamma_i|$.

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A circle not contained in $Z(P)$ meets $Z(P)$ in at most $2D$ points (Bézout Theorem), so each circle intersects $\lesssim D$ cells; hence $\sum_i N_i \lesssim ND$.

Applying the counting lemma inside each cell:

$$|I(\mathcal{S}_{\mathrm{cell}}, \Gamma)| = \sum_i |I(\mathcal{S}_i, \Gamma_i)| \lesssim \sum_i N_i + \sum_i S_i^3 \lesssim ND + (SD^{-2})^2 \sum_i S_i = ND + S^3 D^{-4}.$$

Point-Circle Incidences—Step 3: Algebraic Incidences

Circles that intersect $Z(P)$ (but are not contained in it) contribute at most $2ND$ incidences: $|I(\mathcal{S}_{\text{alg}}, \Gamma_{\text{cell}})| \leq 2ND$. The number of circles fully contained in $Z(P)$ is at most $\frac{D}{2}$.

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Using the counting lemma on points and circles lying on $Z(P)$ gives $|I(\mathcal{S}_{\text{alg}}, \Gamma_{\text{alg}})| \leq S + \frac{D^2}{2}$.

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Using the counting lemma on points and circles lying on $Z(P)$ gives $|I(\mathcal{S}_{\text{alg}}, \Gamma_{\text{alg}})| \leq S + \frac{D^2}{2}$.

If $N > S^3$ or $S > N^2$, counting lemma yields $|I(\mathcal{S}, \Gamma)| \lesssim N$ or $\lesssim S$, respectively. Hence we may restrict to $S^{1/2} \leq N \leq S^3$.

Summing all contributions:

$$|I(\mathcal{S}, \Gamma)| \leq |I(\mathcal{S}_{\text{cell}}, \Gamma)| + \left|I\left(\mathcal{S}_{\text{alg}}, \Gamma_{\text{cell}}\right)\right| + \left|I\left(\mathcal{S}_{\text{alg}}, \Gamma_{\text{alg}}\right)\right| \lesssim ND + S^3 D^{-4} + S + \frac{D^2}{2}.$$

Choose $D \sim S^{3/5} N^{-1/5}$. Consequently,

$$|I(\mathcal{S}, \Gamma)| \lesssim S^{3/5} N^{4/5} + S.$$

Point-Curve Incidences

Now we consider the case for irreducible algebraic curves, which is similar to the case of circles.

Point-Curve Incidences Estimate

Suppose that Γ is a set of N irreducible algebraic curves in the plane of degree at most d. In other words, each curve in Γ is the zero set of an irreducible polynomial $Q \in \mathrm{Poly}_d\left(\mathbb{R}^2\right)$. Setting $k = d^2 + 1$, we have

$$|I(\mathcal{S}, \Gamma)| \lesssim S^{\frac{k}{2k-1}} N^{\frac{2k-2}{2k-1}} + S + N.$$

Point-Curve Incidences (Sketch of Proof)

Counting Lemma (Point-Curve)

If $\mathcal{S}$ is a set of S points in $\mathbb{R}^2$, Γ is a set of N irreducible algebraic curves in $\mathbb{R}^2$, then
(1) $|I(\mathcal{S},\Gamma)| \leq (k-1)N + S^k$;
(2) $|I(\mathcal{S},\Gamma)| \leq (k-1)N^2 + S$.

A curve not contained in $Z(P)$ meets $Z(P)$ in at most dD points (Bézout Theorem), so each circle intersects $\lesssim D$ cells; hence $\sum_i N_i \lesssim ND$.

Applying the counting lemma inside each cell:

$$|I(\mathcal{S}_{cell},\Gamma)| = \sum_i |I(\mathcal{S}_i,\Gamma_i)| \lesssim \sum_i N_i + \sum_i S_i^k \lesssim ND + (SD^{-2})^{k-1} \sum_i S_i \leq ND + S^k D^{-2(k-1)}.$$

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Since every curve intersects $Z(P)$ in at most dD points if it isn't contained in $Z(P)$, we have $|I(\mathcal{S}_{alg},\Gamma_{cell})| \leq NdD$. Consider the degree, the number of curves in $Z(P)$ is at most $\frac{D}{d}$. Using the counting lemma, we also get $|I(\mathcal{S}_{alg},\Gamma_{alg})| \leq S + D^2$.

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Adding them up and we know $|I(\mathcal{S},\Gamma)| \lesssim ND + S^k D^{-2(k-1)} + S + D^2$. Take $D \sim S^{\frac{k}{2k-1}} N^{-\frac{1}{2k-1}}$, therefore, $|I(\mathcal{S},\Gamma)| \lesssim S^{\frac{k}{2k-1}} N^{\frac{2k-2}{2k-1}} + S$.

Constant Degree Partitioning and Harnack Inequality

In the arguments above, we do polynomial partitioning using polynomials of large degree — the degree is typically a polynomial in terms of the size of Γ and the size of $\mathcal{S}$. However, this strategy is no longer effective for the case in $\mathbb{R}^3$ (since we cannot apply Bézout theorem here). Fortunately, there is an alternate method of a slightly weaker estimate which only uses polynomials of a large constant degree. This is a useful technique in some other problems, especially in higher dimensions.

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There's another useful tool to research incidence geometry in $\mathbb{R}^3$.

Theorem (Harnack Inequality)

If $P \in \mathrm{Poly}_D\left(\mathbb{R}^2\right)$, then $\mathbb{R}^2 \backslash Z(P)$ has $\lesssim D^2$ connected components.
Therefore, if $P \in \mathrm{Poly}_D\left(\mathbb{R}^3\right)$, and $\Pi \subset \mathbb{R}^3$ is a 2-plane, then $\Pi \backslash Z(P)$ has $\lesssim D^2$ connected components. In particular, a 2-plane Π enters $\lesssim D^2$ connected components of $\mathbb{R}^3 \backslash Z(P)$.

Point-Plane Incidences—Step 1: Counting Lemma and Making Claim

Point-Plane Incidences Estimate

Suppose that Γ is a set of N 2-planes in $\mathbb{R}^3$ where no three 2-planes are collinear. Suppose that $\mathcal{S}$ is a set of S points in $\mathbb{R}^3$. For any $\varepsilon > 0$, we have

$$|I(\mathcal{S}, \Gamma)| \leq C(\varepsilon)\left(S^{\frac{4}{5}+\varepsilon}N^{\frac{3}{5}+\varepsilon} + S + N\right).$$

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Counting Lemma (Point-Plane)

If $\mathcal{S}$ is a set of S points in $\mathbb{R}^2$, Γ is a set of N 2-planes in $\mathbb{R}^3$ where no three 2-planes are collinear, then: (1) $|I(\mathcal{S}, \Gamma)| \leq 2S^2 + N$; (2) $|I(\mathcal{S}, \Gamma)| \leq \frac{N^3}{2} + 3S$.

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Claim

For any $\varepsilon > 0$, if Γ is a set of N planes in $\mathbb{R}^3$, and $\mathcal{S}$ is a set of S points in $\mathbb{R}^3$, then $|I(\mathcal{S},\Gamma)| \leq C(\varepsilon)S^{\frac{4}{5}+\varepsilon}N^{\frac{3}{5}+\varepsilon} + \left(D^3+3\right)(S+N)$.

Notice that, if the claim holds, by taking $C(\varepsilon) > D^3 + 3$, we have

$$|I(\mathcal{S},\Gamma)| \leq C(\varepsilon)S^{\frac{4}{5}+\varepsilon}N^{\frac{3}{5}+\varepsilon} + (D^3+3)(S+N) \leq C(\varepsilon)\left(S^{\frac{4}{5}+\varepsilon}N^{\frac{3}{5}+\varepsilon} + S + N\right).$$

Point-Plane Incidences—Step 2: Constant Polynomial Partitioning (Twice)

By the polynomial partitioning, we can find a non-zero polynomial P of degree $\leq D$ so that each component of the complement of $Z(P)$ contains $\leq KSD^{-3}$ points of $\mathcal{S}$, where K is a constant. Let kD^3 be the number of cells and O_i be the components of $\mathbb{R}^3 \setminus Z(P)$.

For each i, let $\mathcal{S}_i = \mathcal{S} \cap O_i$ and let $\Gamma_i \subseteq \Gamma$ be the set of planes of Γ that intersect O_i. Let $S_i = |\mathcal{S}_i|$ and $N_i = |\Gamma_i|$. According to the Harnack inequality, every plane intersects $Z(P)$ in at most MD^2 cells, which means $\sum_{i=1}^{kD^3} N_i \leq MND^2$, where M is a constant.

Assume Γ_{alg} is the set of planes of Γ that lie in $Z(P)$ and $\Gamma_{cell} = \Gamma \setminus \Gamma_{alg}$. Let $\mathcal{S}_{alg}$ be the set of points of $\mathcal{S}$ that lie in $Z(P)$ and $\mathcal{S}_{cell} = \mathcal{S} \setminus \mathcal{S}_{alg}$.

Point-Plane Incidences—Step 2: Constant Polynomial Partitioning (Twice)

Let $\mathcal{S}_{alg}^{alg}$ be the set of points of $\mathcal{S}_{alg}$ that lie in $Z(Q)$. Again by polynomial partitioning, we can find a non-zero polynomial Q of degree $\leq D$ which is **co-prime** to P so that each component of the complement of $Z(Q)$ contains $\leq TSD^{-3}$ points of $\mathcal{S}_{alg}$, where T is a constant. (If not, we can consider a small perturbation of Q to make them co-prime).

Let tD^3 be the number of cells and U_i be the components of $\mathbb{R}^3 \setminus Z(Q)$. For each i, let $\mathcal{S}_{alg}^i = \mathcal{S}_{alg} \cap U_i$ and let $\Gamma^i \subseteq \Gamma$ be the set of planes of Γ that intersect U_i. Let $s_i = |\mathcal{S}_{alg}^i|$ and $n_i = |\Gamma^i|$. According to the Harnack inequality, every plane intersects $Z(Q)$ in at most mD^2 cells, which means $\sum_{i=1}^{tD^3} n_i \leq mND^2$, where m is a constant.

Point-Plane Incidences—Step 3: Induction Hypothesis and Inequality (Twice)

Let $p = \frac{4}{5} + \varepsilon$ and $q = \frac{3}{5} + \varepsilon < 1$.

We can observe that the conclusion is established when $S = 1, N = 1$. Assuming that the conclusion is true for $\mathcal{S}', \Gamma'$ such that $|\mathcal{S}'| < |\mathcal{S}|, |\Gamma'| < |\Gamma|$.

Applying the induction hypothesis in each cell and using Jensen's inequality for $\mathcal{S}_{cell}$, we get:

$$\begin{aligned} |I(\mathcal{S}_{cell}, \Gamma)| &= \sum_{i=1}^{kD^3} |I(\mathcal{S}_i, \Gamma_i)| \leq \sum_{i=1}^{kD^3} \left[C(\varepsilon) \left((KSD^{-3})^p N_i^q \right) + \left(D^3 + 3 \right) (S_i + N_i) \right] \\ &\leq C(\varepsilon) K^p D^{-3p} (kD^3) S^p \left(\frac{MND^2}{kD^3} \right)^q + \left(D^3 + 3 \right) \left(S + MD^2 N \right) \\ &= C(\varepsilon) K^p k^{1-p} D^{3-3p-q} M^q S^p N^q + \left(D^3 + 3 \right) (1 + MD^2) S^p N^q \end{aligned}$$

Point-Plane Incidences—Step 3: Induction Hypothesis and Inequality (Twice)

Assume $\mathcal{S}_{alg}^{cell} = \mathcal{S}_{alg} \setminus \mathcal{S}_{alg}^{alg}$. By identically applying the induction hypothesis and Jensen's inequality over the cells U_i generated by polynomial Q, we have:

$$\begin{aligned}|I(\mathcal{S}_{alg}^{cell}, \Gamma)| = \sum_{i=1}^{tD^3} |I(\mathcal{S}_{alg}^{i}, \Gamma^i)| &\leq \sum_{i=1}^{tD^3} \left[C(\varepsilon)\left((TsD^{-3})^p n_i^q\right) + \left(D^3+3\right)\left(s_i+n_i\right)\right] \\ &\leq C(\varepsilon)T^p D^{-3p}(tD^3)S^p \left(\frac{mND^2}{tD^3}\right)^q + \left(D^3+3\right)\left(S+mD^2N\right) \\ &= C(\varepsilon)T^p t^{1-p} D^{3-3p-q} m^q S^p N^q + \left(D^3+3\right)\left(1+mD^2\right)S^p N^q\end{aligned}$$

Adding the bounds for $|I(\mathcal{S}_{cell}, \Gamma)|$ and $|I(\mathcal{S}_{alg}^{cell}, \Gamma)|$ and setting $D = D(\varepsilon)$ and $C(\varepsilon)$ sufficient large, we obtain the intermediate result:

$$|I(\mathcal{S}_{cell}, \Gamma)| + |I(\mathcal{S}_{alg}^{cell}, \Gamma)| \leq C(\varepsilon)S^p N^q$$

Point-Plane Incidences—Step 4: Applying Bézout Theorem and Summing

Consider Bézout Theorem: each plane of Γ has at most D^2 intersection points with $Z(P) \cap Z(Q)$, and so it has at most D^2 incidences with $\mathcal{S}_{alg}$. Thus, we know $|I(\mathcal{S}_{alg}^{alg}, \Gamma_{cell})| \leq ND^2$. Using the initial extreme bounds established earlier, we also have $|I(\mathcal{S}_{alg}^{alg}, \Gamma_{alg})| \leq 3S + \frac{D^3}{2}$.

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Summing these yields $|I(\mathcal{S}_{alg}^{alg}, \Gamma_{cell})| + |I(\mathcal{S}_{alg}^{alg}, \Gamma_{alg})| \leq ND^2 + 3S + \frac{D^3}{2} \leq (D^3 + 3)(S + N)$, since $S, N \geq 1$ and $D \geq 1$.

By combining the cellular and algebraic components, we arrive at the conclusion.

Point-Sphere Incidences

Point-Sphere Incidences Estimate

Suppose that Γ is a set of N 2-spheres in $\mathbb{R}^3$ (of any radii). Note that the intersection of two 2-spheres is always a circle (or a point). Suppose that $\mathcal{S}$ is a set of S points in $\mathbb{R}^3$ with at most 10 on any circle. For any $\varepsilon > 0$, we have

$$|I(\mathcal{S}, \Gamma)| \leq C(\varepsilon) \left(S^{\frac{11}{16}+\varepsilon} N^{\frac{15}{16}+\varepsilon} + S + N \right).$$

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Theorem (Spherical version of the Harnack inequality)

If $P \in \mathrm{Poly}_D\left(\mathbb{R}^3\right)$, and $S^2 \subset \mathbb{R}^3$ is a sphere (with any center or radius), then $S^2 \backslash Z(P)$ has $\lesssim D^2$ connected components.

Counting Lemma (Point-Sphere)

If $\mathcal{S}$ is a set of S points in $\mathbb{R}^2$, Γ is a set of N 2-spheres in $\mathbb{R}^3$ with at most 10 on any circle, then:
(1) $|I(\mathcal{S}, \Gamma)| \leq S^{11} + 55N$; (2) $|I(\mathcal{S}, \Gamma)| \leq 10N^3 + S$.

Claim

For any $\varepsilon > 0$, if Γ is a set of N 2-spheres in $\mathbb{R}^3$, and $\mathcal{S}$ is a set of S points in $\mathbb{R}^3$, then
$I(\mathcal{S}, \Gamma) \leq C(\varepsilon) S^{\frac{11}{16}+\varepsilon} N^{\frac{15}{16}+\varepsilon} + \left(2D^3 + 1\right)(S + N)$.

Final Remark (From Guth's Book)

We have seen how polynomial partitioning gives interesting estimates about a wide variety of incidence problems. However, in all of the estimates we considered today, apart from the Szemerédi-Trotter Theorem, the bounds from polynomial partitioning are not believed to be sharp.

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Thank you!

Any Questions?

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